

Higher Spin Gauge Theories in Any Dimension

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Abstract

Some general properties of higher spin gauge theories are summarized with the emphasize on the nonlinear theories in any dimension.

1 Introduction

First of all, I would like to thank the organizers for the invitation to talk on higher spin gauge theory at Strings 2004. Although a relationship of the higher spin (HS) gauge theory to superstring theory is not yet completely clear, the impressive convergency that took place during recent years indicates that HS theories and Superstring might be different faces of the same fundamental theory to be found. (For an interesting new argument in the same direction see [1] and the talk of M.Bianchi at this conference).

1.1 Symmetric Massless Free Fields

Simplest free HS gauge fields are so called totally symmetric fields. They can be described by rank s totally symmetric tensors $\varphi_{n_1 \dots n_s}(x)$ subject to the double-tracelessness condition $\varphi_{m \dots k n_5 \dots n_s}(x) = 0$ [2] ($m, n, \dots = 0, \dots, d-1$). The Abelian HS gauge symmetries $\delta\varphi_{n_1 \dots n_s}(x) = \partial_{\{n_1} \varepsilon_{n_2 \dots n_s\}}(x)$ with rank $s-1$ symmetric traceless gauge parameters $\varepsilon_{n_1 \dots n_{s-1}}(x)$ ($\varepsilon^r{}_{rn_3 \dots n_{s-1}}(x) = 0$) leave invariant the quadratic action

$$\begin{aligned}
 S^s = & \frac{1}{2}(-1)^s \int d^d x \{ \partial_n \varphi_{m_1 \dots m_s} \partial^n \varphi^{m_1 \dots m_s} \\
 - & \frac{s(s-1)}{2} \partial_n \varphi^r{}_{rm_1 \dots m_{s-2}} \partial^n \varphi^k{}_{kn_5 \dots n_s} + s(s-1) \partial_n \varphi^r{}_{rm_1 \dots m_{s-2}} \partial_k \varphi^{nkm_1 \dots m_{s-2}} \\
 - & s \partial_n \varphi^n{}_{m_1 \dots m_{s-1}} \partial_r \varphi^{rm_1 \dots m_{s-1}} - \frac{s(s-1)(s-2)}{4} \partial_n \varphi^r{}_{rm_1 \dots m_{s-3}} \partial_k \varphi^t{}_{tkm_1 \dots m_{s-3}} \}.
 \end{aligned} \tag{1}$$

This action describes a spin s massless field [2] and generalizes the spin 1 Maxwell action and spin 2 (linearized) Einstein action to any integer spin. The key question is what is a fundamental unifying symmetry principle underlying HS gauge fields. Even at the free field level, the existence of elegant metric-like [3] and frame-like [4, 5] “geometric formulations” indicates that there must be some deep reason for HS theories to exist.

1.2 Higher Spin Problem

The problem is to find a nonlinear HS gauge theory such that it has

- Correct free field limit
- Unbroken HS gauge symmetries
- Non-Abelian global HS symmetry of a vacuum solution

The first condition demands the theory to be free of ghosts, that is to be equivalent to the Fronsdal theory for the case of totally symmetric fields. The third condition avoids trivial possibility of Abelian interactions built of Abelian gauge invariant HS field strengths like nonlinear terms built of higher powers of the spin 1 Abelian field strength instead of Yang-Mills interactions.

The HS problem is of interest on its own right. An additional stringy motivation is, in the first place, that it is tempting to interpret massive HS modes in Superstring as resulting from breaking of HS gauge symmetries. In that case, superstring should exhibit higher symmetries in the high-energy limit as was argued long ago by Gross [6]. A more recent argument came from the *AdS/CFT* side after it was realized [7] that HS symmetries should be unbroken in the Sundborg–Witten limit $\lambda = g^2 N \rightarrow 0$, $l_{str}^2 \Lambda_{AdS} \rightarrow \infty$ just because the boundary conformal theory becomes free. A dual string theory in the highly curved *AdS* space-time is therefore going to be a HS theory.

1.3 Difficulties

Although the formulation of the HS problem may look rather flexible, the conditions on the HS interactions is so hard to satisfy that many believed it admits no solution at all. One difficulty is due to the *S*-matrix argument *a la* Coleman-Mandula [8] that, if *S*-matrix has too many symmetries carrying

nontrivial representations of the Lorentz symmetry as HS symmetries do, then $S = I$, i.e. there is no interactions.

Another one is the HS-gravity interaction problem as was originally pointed out by Aragone and Deser in [9]. The point is that covariantization of derivatives $\partial_n \rightarrow D_n = \partial_n - \Gamma_n$, $\delta\varphi_{nm\dots} \rightarrow D_n \varepsilon_{m\dots}$ changes the situation drastically because they do not commute, $[D_n, D_m] = \mathcal{R}_{nm}\dots$, if the Riemann tensor $\mathcal{R}_{nm,pq}$ is nonzero. As a result, the variation of the covariantized HS action under covariantized HS gauge theories is not any longer zero

$$\delta S_s^{cov} = \int \mathcal{R}\dots(\varepsilon\dots D\varphi\dots) \neq 0 \quad ?! \quad (2)$$

Most important is that the Weyl tensor part of the Riemann tensor contributes to this variation for $s > 2$, which contribution seems to be hard to compensate by any modification of the action and/or field transformations¹.

1.4 Resolution

Despite the difficulties with HS interactions, in the important works [10, 11] it was shown that some consistent (i.e., gauge invariant) interactions of HS gauge fields with matter fields and with themselves do exist at least at the cubic level

$$S = S^2 + S^3 + \dots \quad S^3 = \sum_{p,q,r} (D^p \varphi)(D^q \varphi)(D^r \varphi) \ell^{p+q+r+\frac{1}{2}d-3}, \quad (3)$$

where ℓ is a parameter of dimension of length. These authors discovered that interactions of HS fields contain higher space-time derivatives: the higher interacting spins are, the more (but finite) number of derivatives appear in their interactions.

Another important observation was [12] that the situation with HS interactions and, in particular, with HS-gravitational interactions, improves once the problem is reconsidered in the $(A)dS$ background. The dimensionful parameter ℓ then identifies with the radius of $(A)dS$ space $\ell = \Lambda^{-\frac{1}{2}} = R_{AdS}$. The key difference between flat and $(A)dS$ background is that, in the latter case, background covariant derivatives do not commute $[D_n, D_m] \sim \Lambda \sim O(1)$. This has

¹Recall that, for spin 3/2, analogous terms can be compensated by the modification of the transformation of the metric tensor under local SUSY transformation because only Ricci tensor contributes in this case, that opens a way towards supergravity.

an important consequence that the terms of different orders in derivatives in the action (3) do talk to each other. As a result, there exists [12] such a unique (modulo field redefinitions) combination of higher derivative interaction terms that a contribution from their gauge variation cancels the problematic terms (2) of the flat space analysis. Since the coupling constants of the interaction terms contain positive powers of the $(A)dS$ radius, they blow up in the flat limit in agreement with the flat space no-go results. In $(A)dS$ space, the no-go arguments do not work (in particular, no S -matrix). HS theories suggest deep analogy between the AdS scale and string length scale $\ell \sim \sqrt{\alpha'}$, although the precise identification is far from being clear at the moment. Note that the role of $(A)dS$ background in HS theories was realized years before the discovery of the AdS/CFT correspondence [13].

2 HS Fields as Gauge Connections

It is well-known that gauge fields of supergravity result from gauging the SUSY algebra:

$$\begin{array}{llll}
o(d-1, 2) & & o(N) & \\
T^{AB} & Q_\alpha^p & t^{pq} & A, B, \dots = 0, \dots d, \quad p, q, \dots = 1, \dots N \\
\omega_n^{AB} & \Psi_{n\alpha}^p & A_n^{pq} & n = 0, \dots d-1 \quad \alpha \text{ is spinor index}
\end{array}$$

In particular, $s = 2$ gravitational field is described in supergravity by the frame field e_n^a which along with the Lorentz connection ω_n^{ab} can be interpreted [14] as components of the gauge field ω_n^{AB} of the AdS_d algebra $o(d-1, 2)$

$$g_{nm} \longrightarrow e_n^a \longrightarrow \{e_n^a, \omega_n^{ab}\} \longrightarrow \omega_n^{AB} \quad \boxed{\begin{smallmatrix} & \\ & \end{smallmatrix}}$$

Analogously, a totally symmetric field $\varphi_{n_1 \dots n_s}$ in the Fronsdal formulation admits an equivalent description in terms of the gauge 1-form $\omega_n^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ [5]

$$\varphi_{n_1 \dots n_s} \longrightarrow e_n^{a_1 \dots a_{s-1}} \longrightarrow \{e_n^{a_1 \dots a_{s-1}}, \omega_n^{a_1 \dots a_{s-1}, b_1 \dots b_{s-1}}\} \longrightarrow \omega_n^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}, \quad (4)$$

which takes values in the irreducible representation of the AdS_d algebra $o(d-1, 2)$ depicted by the (traceless) rectangular two-row Young tableau

$$\begin{array}{ll}
\omega_n^{\{A_1 \dots A_{s-1}, A_s\} B_2 \dots B_{s-1}} = 0 & \begin{array}{c} s-1 \\ \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \end{array} \\
\omega_n^{A_1 \dots A_{s-3} C}{}_{C,}{}^{B_1 \dots B_{s-1}} = 0 & o(d-1, 2)
\end{array}$$

Let an AdS vector V_A define the Lorentz subalgebra $o(d-1, 1) \subset o(d-1, 2)$ as its stability subalgebra. The simplest choice is $V_B = \delta_B^{\hat{d}}$ where \hat{d} denotes the $(d+1)^{th}$ Lorentz invariant direction of an AdS vector. The HS dynamical frame-like field is then identified with the components of the HS connection with a maximal possible number of AdS_d vector components along the extra direction V_A

$$e_n^{a_1 \dots a_{s-1}} = \omega_n^{a_1 \dots a_{s-1}, B_1 \dots B_{s-1}} V_{B_1} \dots V_{B_{s-1}}. \quad (5)$$

Analogously to the spin 2 metric case, Fronsdal field is the totally symmetric part of the frame field $\varphi_{n_1 n_2 \dots n_s} = e_{\{n_1, n_2 \dots n_s\}}$. Generalized Lorentz connections identify with those components of the connection that carry more Lorentz indices

$$\omega_n^{a_1 \dots a_{s-1}, b_1 \dots b_t} = \omega_n^{a_1 \dots a_{s-1}, b_1 \dots b_t B_{t+1} \dots B_{s-1}} V_{B_{t+1}} \dots V_{B_{s-1}}. \quad (6)$$

Upon resolving appropriate torsion-like constraints [5], the generalized Lorentz connections are expressed through derivatives of the frame-like field

$\omega_n^{a_1 \dots a_{s-1}, b_1 \dots b_t} \sim \left(\frac{1}{\sqrt{\Lambda}} \frac{\partial}{\partial x} \right)^t (e)$ so that every additional Lorentz index brings one derivative along with one power of $\ell = \frac{1}{\sqrt{\Lambda}}$. In this formalism, the higher derivatives of HS interactions, as well as negative powers of the cosmological constant, result from the dependence of the nonlinear terms in the HS actions on the higher generalized Lorentz connections.

3 Higher Spin Algebra $hu(1|2; [d-1, 2])$

The structure of HS gauge connections suggests that they result from gauging a HS algebra that contains the AdS_d algebra $o(d-1, 2)$ as a subalgebra and decomposes under adjoint action of the latter into a sum of representations described by traceless two-row Young tableaux. In other words, generators $T_{A_1 \dots A_n, B_1 \dots B_n}$ of a HS algebra should satisfy the conditions $T_{\{A_1 \dots A_n, A_{n+1}\} B_2 \dots B_n} = 0$ and $T^C_{C A_3 \dots A_n, B_1 \dots B_n} = 0$. Such an algebra was originally found by Eastwood [17] as conformal HS algebra of a scalar field in $d-1$ dimensions. For our purpose it is most convenient to use its oscillator realization suggested in [18].

Namely we introduce a canonical pair of AdS vectors Y_i^A ,

$$[Y_i^A, Y_j^B]_* = \epsilon_{ij} \eta^{AB}, \quad (7)$$

where $\eta_{AB} = \eta_{BA}$ is the AdS_d invariant metric and $\epsilon_{ij} = -\epsilon_{ji}$ $i, j = 1, 2$ is the $sp(2)$ invariant form. Here we use the star product notation for the

oscillator algebra defined by the relations (7) (for its precise definition see eq.(25) for Z -independent functions). The bilinear combinations of oscillators, T^{AB} and t_{ij} ,

$$T^{A,B} = -T^{B,A} = \frac{1}{2}\{Y_i^A, Y_j^B\}_* \epsilon^{ji}, \quad t_{ij} = t_{ji} = \frac{1}{2}\{Y_i^A, Y_{jA}\}_*, \quad (8)$$

form, respectively, the $o(d-1, 2)$ generators, which rotate AdS_d vector indices A, B , and $sp(2)$ generators, which rotate symplectic indices i, j . They commute to each other, $[t_{ij}, T^{AB}]_* = 0$, thus being Howe dual. Let us note that the $sp(2)$ plays here a role analogous to that in the description of dynamical models in the conformal framework (two-time physics) [15, 16].

Now it is easy to define the simplest HS algebra $hu(1|2:[d-1, 2])$ by virtue of a sort of Hamiltonian reduction. First one considers the Lie algebra of functions of oscillators with the Lie bracket $[f(Y), g(Y)]_*$. Then one considers its subalgebra S spanned by $sp(2)$ invariants

$$[f(Y), t_{ij}]_* = 0 \quad (9)$$

and next its quotient S/I where the ideal I is spanned by the elements proportional to the $sp(2)$ generators i.e., $\{f \in I: f(Y) = t^{ij} * f_{ij} = f_{rij} * t^{ij} \sim 0.\}$ The algebra S/I we call $hu(1|2:[d-1, 2])$ (upon imposing appropriate reality conditions [18]).

The gauge fields of $hu(1|2:[d-1, 2])$ are

$$\omega(Y|x) = \sum_{n=0}^{\infty} dx^m \omega_{m A_1 \dots A_n, B_1 \dots B_n}(x) Y_1^{A_1} \dots Y_1^{A_n} Y_2^{B_1} \dots Y_2^{B_n}. \quad (10)$$

It is easy to see that the condition (9) (which can be written in the covariant form $Dt_{ij} = dt_{ij} + [\omega, t_{ij}]_* = 0$ taking into account $dt_{ij} = 0$) imposes the Young properties

$$\omega_{m \{A_1 \dots A_n, A_{n+1}\} B_2 \dots B_n} = 0 \quad \begin{array}{c} n \\ \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} \end{array}$$

(including the property that the r.h.s. of (10) contains equal numbers of oscillators Y_1^A and Y_2^A). According to (8), the factorization over the terms proportional to t_{ij} is equivalent to factorization of traces of the gauge field components in (10) that gives rise precisely to the set of gauge fields associated with different spins as explained in section 2.

The non-Abelian HS field strength is

$$R = d\omega(Y|x) + \omega(Y|x) \wedge * \omega(Y|x), \quad (11)$$

where terms on the r.h.s., which take values in the ideal I , are factored out. The infinite-dimensional HS algebra contains the maximal finite-dimensional subalgebra $o(d-1, 2) \oplus u(1)$ spanned by bilinears in oscillators and constants, respectively. Different spins correspond to homogeneous polynomials $\omega(\mu Y|x) = \mu^{2(s-1)} \omega(Y|x)$. The gauge fields of $o(d-1, 2) \oplus u(1)$ carry spin 2 and spin 1 respectively. That $o(d-1, 2) \oplus u(1)$ is a maximal finite-dimensional subalgebra of $hu(1|2; [d-1, 2])$ is a consequence of the fact that the commutator of degree p and degree q polynomials of oscillators gives a degree $p+q-2$ polynomial. For example, if spin 3 associated with degree 4 polynomials in oscillators appears, polynomials of all higher degrees appear in the closure of its generators. Thus, beyond the barrier of spin 2, the systems of HS fields are necessarily infinite. Let us note that there exists a generalization of the nonlinear HS gauge theory to the case with HS gauge connection carrying matrix indices $\omega \rightarrow \omega_q^p(Y|x)$ $p, q = 1, \dots, n$ so that the spin 1 Yang-Mills algebra is promoted to $u(n)$ (HS models with the Yang-Mills gauge algebras $o(n)$ and $usp(n)$ also exist [18]).

4 Lower Spin Examples

To illustrate the idea of the approach that allows us to formulate nonlinear HS dynamics let us start with lower spin examples.

The nonlinear $s = 2$ equations are equivalent to zero-torsion condition $R^a = 0$ together with the Einstein equation in the form

$$R^{ab} = e_c \wedge e_d C^{ac, bd}, \quad (12)$$

where e is the frame 1-form and $C^{ac, bd}$ has algebraic properties of the Weyl tensor, i.e. it is traceless and symmetrization over three indices gives zero². The equation (12) tells us that $C^{ac, db}$ is indeed the Weyl tensor and, because it is traceless, that the Ricci tensor is zero. Bianchi identities then imply at

²For the future convenience we use symmetric basis with $C^{ac, bd} = C^{ca, bd} = C^{ac, db}$. The relationship with the standard antisymmetric Weyl tensor $\tilde{C}^{[ac], [bd]}$ is $C^{ab, cd} = \frac{1}{2} (\tilde{C}^{[ac], [bd]} + \tilde{C}^{[bc], [ad]})$.

the linearized level that non-zero components of order k derivatives of the Weyl tensor $\partial_{n_1} \dots \partial_{n_k} C_{a_1 a_2, b_1 b_2}$ form Lorentz tensors $C_{c_1 \dots c_{k+2}, d_1 d_2}$ described

by the Young tableaux $\begin{array}{c} k+2 \\ \square \square \square \square \square \square \end{array}$. Einstein equations imply that all $C_{a_1 \dots a_{k+2}, b_1 b_2}$ are traceless.

In terms of the quantities $C_{c_1 \dots c_{k+2}, d_1 d_2}$, consequences of the linearized $s = 2$ equations in flat space can be written in the form of covariant constancy conditions

$$dC_{a_1 \dots a_l, b_1 b_2} = e_0^c \left(l C_{a_1 \dots a_l c, b_1 b_2} + 2 C_{a_1 \dots a_l \{b_1, b_2\} c} \right), \quad (13)$$

where e_0^a is the flat Minkowski frame $e_0^a = dx^a$.

Analogously, $s = 1$ Maxwell equations can be reformulated as

$$F = e_0^c \wedge e_0^d C_{c, d}, \quad (14)$$

$$dC_{a_1 \dots a_l, b} = e_0^c \left((l+1) C_{a_1 \dots a_l c, b} + C_{a_1 \dots a_l b, c} \right) \quad (15)$$

with the 0-forms $C_{a_1 \dots a_l, b}$ described by the traceless Young tableaux $\begin{array}{c} k+1 \\ \square \square \square \square \square \end{array}$

$s = 0$ Klein-Gordon dynamics is reformulated in the form

$$dC_{a_1 \dots a_k} = e_0^c (k+2) C_{a_1 \dots a_k c} \quad (16)$$

in terms of symmetric traceless 0-forms $C_{a_1 \dots a_k}$, which parametrize all on-mass-shell nontrivial combination of derivatives of the scalar field C and are described by the Young tableaux $\begin{array}{c} k \\ \square \square \square \square \square \end{array}$.

This formulation extends naturally to any spin s described by “Weyl 0-forms” $C_{a_1 \dots a_{s+k}, b_1 \dots b_s}$ with the symmetry properties of the traceless Young

tableaux $\begin{array}{c} s+k \\ \square \square \square \square \square \square \square \end{array}$. The meaning of the set of 0-forms $C_{a_1 \dots a_{s+k}, b_1 \dots b_s}$

is that they form a basis in the space of gauge invariant on-mass-shell nontrivial derivatives of a massless field under consideration. As a result, the space of $C_{a_1 \dots a_{s+k}, b_1 \dots b_s}(x)$ at any given x is analogous (in fact, dual by a nonunitary Bogolyubov transform) to the space \mathcal{H} of spin s single-particle states. Thus, Weyl 0-forms are sections of the fiber bundle over space-time with the fiber space dual of the space of single-particle quantum states in the system.

5 Central On-Mass-Shell Theorem

The next step is to observe that free massless field equations in $(A)dS_d$ space can be concisely formulated in terms of the star product algebra. To this end one describes the background gravitational field as flat connection $\omega_0 \in o(d-1, 2)$, $R^{AB}(\omega_0) = 0$ with $\omega_0^{AB} \neq 0$, and $\omega_0^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} = 0$ for $s > 2$. In the linearized approximation one sets $\omega(Y|x) = \omega_0(Y|x) + \omega_1(Y|x)$ where $\omega_1(Y|x)$ describes dynamical fluctuations. Then the generalization of the free lower spin equations (12)-(16) to the free equations for massless fields of all spins (plus constraints on auxiliary fields) is [5, 18]

$$R_1(Y|x) = \frac{1}{2} e_0^a \wedge e_0^b \frac{\partial^2}{\partial Y_i^a \partial Y_j^b} \varepsilon_{ij} C(Y|x) \Big|_{V_A Y_i^A = 0}, \quad (17)$$

$$\widetilde{D}_0(C) = 0, \quad t_{ij} * C = C * t_{ij}, \quad D(t_{ij}) = 0, \quad (18)$$

where R_1 is the linearized HS field strength (11) and $\widetilde{D}_0(C)$ is the covariant derivative in the twisted adjoint representation,

$$R_1 = d\omega + \omega_0 * \omega_1 + \omega_1 * \omega_0, \quad \widetilde{D}_0(C) = dC + \omega_0 * C - C * \widetilde{\omega}_0, \\ \widetilde{f}(Y) = f(\widetilde{Y}), \quad \widetilde{Y}_i^A = Y_i^A - \frac{2}{V^2} V^A V_B Y_i^B. \quad (19)$$

6 Nonlinear Construction

6.1 General idea

The form of the equations (17), (18) suggests the idea [19] to search nonlinear HS equations in the “unfolded” form of generalized flatness conditions

$$d\omega^\Phi = F^\Phi(\omega) \quad d = dx^n \frac{\partial}{\partial x^n}, \quad (20)$$

where $\omega^\Phi(x)$ is a set of differential forms (including 0-forms) and the function $F^\Phi(\omega)$ contains only wedge products of $\omega^\Phi(x)$ and is such that the consistency condition

$$F^\Phi \wedge \frac{\delta F^\Omega}{\delta \omega^\Phi} = 0 \quad (21)$$

is true for any $\omega^\Phi(x)$. (Once this is the case, the function $F^\Phi(\omega)$ defines a free differential algebra [20]).

The unfolded form (20) of the field equations has several nice properties:

- It is manifestly invariant under gauge transformations

$$\delta\Omega^\Phi = d\varepsilon^\Phi - \varepsilon^\Omega \frac{\delta F^\Phi}{\delta\omega^\Omega}, \quad \deg \varepsilon^\Phi(x) = \deg \omega^\Phi(x) - 1. \quad (22)$$

- Invariant under diffeomorphisms.
- Interactions: a nonlinear deformation of $F^\Phi(\omega)$.
- Degrees of freedom are in the 0-form fields which form an infinite-dimensional module dual to the space of single-particle states.
- Universality: any dynamical system can be reformulated in the unfolded form.

Originally it was shown by direct inspection that a nonlinear deformation of the $d = 4$ unfolded free massless field equations (18) exists in the lowest orders [19]. To go beyond lowest orders some more sophisticated approach was needed. The useful idea was [22] to find an appropriate generalization g' of the HS algebra g such that a substitution

$$\omega \rightarrow W = \omega + \omega C + \omega C^2 + \dots \quad (23)$$

into the g' zero curvature equation $dW + W \wedge W = 0$ reconstructs nonlinear HS equations. The key issue is of course to find restrictions on W that reconstruct (23) in all orders. The guiding principle is [21, 18] to preserve $sp(2)$ at the nonlinear level. Before going into details of the construction let us mention that the resulting interactions are unique up to field redefinitions. The only dimensionless coupling constant is the YM constant $g^2 = |\Lambda|^{\frac{d-2}{2}} \kappa^2$ which, however, is artificial in the classical pure gauge HS theory because it can be rescaled away just as in the classical pure Yang-Mills theory.

6.2 Nonlinear HS Equations

In [18] it was shown that the appropriate extension $g \rightarrow g'$ is achieved by the doubling of oscillators $Y_i^A \rightarrow (Z_i^A, Y_i^A)$ so that the HS fields extend to $\omega(Y|x) \rightarrow W(Z, Y|x)$, $C(Y|x) \rightarrow B(Z, Y|x)$. In addition we introduce the S connection along Z_i^A that together with W form a noncommutative connection

$$\mathcal{W} = d + W + S, \quad S(Z, Y|x) = dZ_i^A S_A^i. \quad (24)$$

The star product in g' is

$$(f * g)(Z, Y) = \int dS dT f(Z + S, Y + S) g(Z - T, Y + T) \exp 2S_A^i T_i^A. \quad (25)$$

One can see that this is the oscillator algebra with the nonzero basis relations $[Y_i^A, Y_j^B]_* = -[Z_i^A, Z_j^B]_* = \varepsilon_{ij} \eta^{AB}$. It is not however the Moyal star product, being the normal ordered star product with respect to $Z - Y : Z + Y$ normal ordering because the left star multiplication by $Z - Y$ and right star multiplication by $Z + Y$ are equivalent to the usual pointwise multiplication.

An important property of this star product is that it admits the Klein operator \mathcal{K} that generates the automorphism (19)

$$\mathcal{K} = \exp \frac{2}{V^2} V_A Z^{Ai} V_B Y_i^B, \quad \mathcal{K} * f = \tilde{f} * \mathcal{K}, \quad \mathcal{K} * \mathcal{K} = 1. \quad (26)$$

The nonlinear HS field equations can be concisely formulated in the form [18]

$$\mathcal{W} * \mathcal{W} = \frac{1}{2} (dZ_A^i dZ_i^A + 4\Lambda^{-1} dz^i dz_i B * \mathcal{K}) \quad \mathcal{W} * B = B * \tilde{\mathcal{W}}, \quad (27)$$

where $\tilde{\mathcal{W}}(dZ, Z, Y) = \mathcal{W}(\tilde{dZ}, \tilde{Z}, \tilde{Y})$ and $dz_i = \frac{1}{\sqrt{V^2}} V_B dZ_i^B$. This system is manifestly gauge invariant under the gauge transformations

$$\delta \mathcal{W} = [\varepsilon, \mathcal{W}]_*, \quad \delta B = \varepsilon * B - B * \tilde{\varepsilon} \quad (28)$$

with an arbitrary gauge parameter $\varepsilon = \varepsilon(Z, Y|x)$. One of the most important properties of the system (28) is [18] that it admits $sp(2)$ such that its generators τ_{ij} form a nonlinear deformation of the $sp(2)$ algebra (8) and single out the physical sector of HS fields by the $sp(2)$ invariance condition $D\tau_{ij} = 0$ (followed by factorization of the terms proportional to τ_{ij}). The nonlinearly realized $sp(2)$ can be interpreted as a symmetry of a two-dimensional fuzzy hyperboloid in the noncommutative space of Y_i^A and Z_j^A . A radius of the fuzzy hyperboloid depends on $B(Z, Y|x)$ which is the generating function for the Weyl 0-forms.

To analyse the HS field equations perturbatively one sets

$$W = W_0 + W_1, \quad S = dZ_i^A Z_A^i + S_1, \quad B = B_1, \quad (29)$$

where $W_0 = \frac{1}{2} \omega_0^{AB}(x) Y_A^i Y_{iB}$ with $\omega_0^{AB}(x)$ describing background AdS_d gravitational field. It is not hard to see that central on-mass-shell theorem (18) is reproduced in the lowest order [18].

Let us note that the form of the noncommutative connection S in (29) implies that, because of the first term in S , the HS symmetry (28) is spontaneously broken down to the HS symmetry with Z -independent parameters $\varepsilon(Y|x)$ (The Z -dependent components in $\varepsilon(Z, Y|x)$ are used to gauge fix the noncommutative Z -connection S). Because of the B -dependent term in (28), the leftover HS symmetries with the HS gauge parameters $\varepsilon(Y|x)$ acquire B -dependent nonlinear corrections. As a result, HS gauge symmetries in the nonlinear HS theory are different from the Yang-Mills gauging of the global HS symmetry of a free theory one starts with.

7 Singletons in any Dimension

The simplest HS algebra $hu(1|2:[d-1, 2])$ admits the fermionic generalization $hu(1|(1, 2):[d-1, 2])$

$$sp(2) \rightarrow osp(1, 2) \quad Y_i^A \rightarrow (Y_i^A, \phi^A), \quad \{\phi^A, \phi^B\} = -2\eta^{AB}. \quad (30)$$

The Fock-type modules of $hu(1|2:[M, 2])$ and $hu(1|(1, 2):[M, 2])$ describe massless scalar S_M and spinor F_M in M dimensions [23] (closely related analysis of M -dimensional field equations in terms of conformal algebra and dual $sp(2)$ and $osp(1, 2)$ was given in [15, 16]). This gives realization of the HS algebras as conformal HS algebras acting on the scalar and spinor conformal fields (i.e., singletons) in M dimensions.

The following generalization of the 4d Flato-Fronsdal theorem [24] takes place [23]:

$$S_{d-1} \otimes S_{d-1} = \sum_{s=0}^{\infty} \oplus \overbrace{\square \square \square \square \square \square \square}^s \quad m=0 \quad \text{bosons in } AdS_d \quad (31)$$

(note that related statements were discussed in [25])

$$F_{d-1} \otimes S_{d-1} = \sum_{s=\frac{1}{2}}^{\infty} \oplus \overbrace{\square \square \square \square \square \square \square}^{s-1/2} \quad m=0 \quad \text{fermions in } AdS_d \quad (32)$$

$$F_{d-1} \otimes F_{d-1} = \sum_{p,q} \oplus \overbrace{\begin{array}{c} \square \square \square \square \square \square \square \\ \square \\ \square \\ \square \end{array}}^{p+1} \quad m=0 \quad \text{bosons in } AdS_d$$

$$\oplus m > 0 \quad \text{antisymmetric tensors} \quad (33)$$

These results show precise matching between spectra of gauge fields in HS theories and appropriate UIRs of HS algebras, indicating the existence of HS gauge theories with fermions and mixed symmetry fields [23]. Moreover, HS superalgebras exist in any d [23]. There is no contradiction here with the absence of usual supersymmetries in higher dimensions because HS superalgebras contain usual finite-dimensional subsuperalgebras only for some lower dimensions like $d = 3, 4, 5$.

8 Conclusions

The main conclusion is that nonlinear HS theories exist in any dimension. Note that HS gauge symmetries in the nonlinear HS theory differ from the Yang-Mills gauging of the global HS symmetry of a free theory one starts with by HS field strength dependent nonlinear corrections resulting from the partial gauge fixing of spontaneously broken HS symmetries in the extended noncommutative space.

The HS geometry is that of the fuzzy hyperboloid in the auxiliary (fiber) noncommutative space. Its radius depends on the Weyl 0-forms which take values in the infinite-dimensional module dual to the space of single-particle states in the system.

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